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New Models for the Solution of Intermediate Regimes in Transport Theory and Radiative Transfer: Existence Theory, Positivity, Asymptotic Analysis, and Approximations

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In this paper we propose, derive, and establish the mathematical foundations of new models for the solution of intermediate regimes in transport theory and radiative transfer. These new models consist of coupling the transport equations with their diffusion approximations. Our mathematical theory includes the rigorous derivation of these models, the existence theory, the positivity of the solutions, and the asymptotic analysis. We also give the rate of the asymptotic decay. In order to solve the new coupled problem we propose to use the transmission time marching algorithm introduced and studied in refs. 10, 13–15. We then study the convergence of the resulting algorithm. These studies are based in an essential way on the methods we introduced in refs. 14, 15.

KEY WORDS: Asymptotic behaviour; boundary layers correctors coupling of transport equations and their diffusion approximations; diffusion equations; transmission time marching algorithm; transport equations.

1. INTRODUCTION

The scaled transport equations [1, 2, 3] correspond to finding $u_\epsilon(x, v, t)$ such that

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon} v \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon^2} \Sigma (K - I)\, u_\varepsilon + \gamma u_\varepsilon \qquad \text{in} \quad X \times S^2 \times\,]0, +\infty[\tag{1}$$

$$u_\varepsilon(x, v, t) = 0, \qquad (x, v) \in \Gamma_-, \quad t > 0 \tag{2}$$

$$u_\varepsilon(x, v, 0) = u_I(x) \qquad \text{in } X \times S^2 \tag{3}$$

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where X is an open bounded of $\mathbb{R}^3$, $\Gamma_- = \{(x, v) \in \partial X \times S^2; v_x \cdot v < 0\}$, S^2 is the unit sphere of $\mathbb{R}^3$, $(Kg)(x, v) = \int_{S^2} f(v', v)\, g(v')\, dv'$ $\forall g \in L^2(S^2)$, $u_\varepsilon(x, v, t)$ is the density of neutrons at x with velocity v at time t, $u_I = u_I(x)$ is a given positive function independent of v. The data Σ is a bounded and positive function depending only of the position x. The data γ is a bounded function. The parameter ε is the mean free path, which is defined as the ratio of the average distance traveled by a neutron between two successive collisions and a characteristic length of the problem in consideration. The function f is defined by $f(x, v', v) = |v'|\, c(x, v')\, g(x, v' \to v)$ with

$g(x, v' \to v)\, dv$ probability that any secondary neutron (which may be the original neutron but with a new velocity in a simple scattering event) induced by an incident neutron with velocity v' will be emitted with velocity v in dv

$c(x, v')$ mean number of secondary neutrons emitted in a collision event experienced by an incident particle with velocity v at position x

This equation describes the evolution of a neutrons population in a domain of $\mathbb{R}^3$ occupied by a medium which is in interaction with the neutrons. Similar processes arise in wide variety of physical phenomena such as radiative transfer, etc...

Solving the transport equations is difficult and this difficulty increases as the mean free path becomes smaller. Therefore it is apparent that the development of approximate descriptions to the transport equations is a very important aspect of transport theory. One of the most important approximations consists of approximations that remove the velocity dependence of the transport equation. This leads for example in neutron transport or radiative transfer to the so-called diffusion approximations. We shall first describe the derivation of such equations and then examine their domain of validity.

Assuming for simplicity that u_I is independent of v, we then obtain:[6]

The solution u_ε of the transport problem

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon}\, v \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon^2}\, \Sigma(K - I)\, u_\varepsilon + \gamma u_\varepsilon \qquad \text{in} \quad X \times S^2 \times]0, +\infty[\tag{4}$$

$$u_\varepsilon(x, v, t) = 0, \qquad (x, v) \in \Gamma_-, \quad t > 0 \tag{5}$$

$$u_\varepsilon(x, v, 0) = u_I(x) \qquad \text{in} \;\; X \times S^2 \tag{6}$$

and the solution u of the diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + \gamma u \qquad \text{in } X \times]0, +\infty[\tag{7}$$

$$u(x, t) = 0, \qquad x \in \partial X, \quad t > 0 \tag{8}$$

$$u(x, 0) = u_I(x), \qquad \text{in } X \tag{9}$$

satisfy

$$\forall t \geqslant 0 \; \|u_\varepsilon(\cdot, \cdot, t) - u(\cdot, t)\|_{L^\infty(X \times S^2)} \leqslant \varepsilon e^{\delta t(1+t)} C_{u_I}$$

where $\delta = \sup_x \gamma(x)$, and C_{u_I} is a positive constant (independent of ε). $L^\infty(X \times S^2)$ denotes the space of bounded measurable functions in $X \times S^2$.

This approximation is valid under the additional assumption:

The point x is far from the boundary ∂X and far from the regions of X for which the data Σ, f, and q have large variation.

Results of this type are obtained by various authors; see for example refs. 8, 9, 11, 3, 2, 12, 1 and the references therein.

The coefficients of the diffusion equations can be computed from those of the transport equations. The diffusion equations (7)–(9) are much easier to study than the transport equations (4)–(6):

On the mathematical level, diffusion equations are endowed with rich mathematical theory,

On the numerical level the diffusion equations are much easier to solve on computer than the transport equations.

In a number of applications we use the transport equations to compute the physical coefficients involved in the diffusion equations, and then we solve the diffusion equations.

A second order approximation in ε of the solution u_ε of the transport equations is obtained in the following result (see ref. 6 and the references therein for more details)

Theorem 1.1. Let u_ε be the solution of Eqs. (4)–(6). Let u and w be the solutions of

$$-\frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) - \gamma u = g, \qquad \text{in } X,$$

$$u|_{\partial X} = 0$$

$$-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right)-\gamma w=0, \qquad \text{in } X$$

$$w|_{\partial X}=-\frac{L}{\Sigma}\frac{\partial u}{\partial v}$$

where L is a positive constant (independent of the data g); then we have for all spaces $L^p(1\leqslant p<\infty)$

$$\left\|u_\varepsilon-\left[u+\varepsilon\left(-D_i\frac{1}{\Sigma}\frac{\partial u}{\partial x_i}+w+b\right)\right]\right\|_{L^p(X\times V)}\leqslant \varepsilon^2 C_{p,g}$$

where b is a boundary layer corrector, $C_{p,g}$ is a constant. L^p is the space of measurable functions such that $\int_X |g|^p\,dx$ is finite.

The introduction of the boundary layer correctors which are related to the Milne problem implies that for the free surface boundary condition the density is approximated by 0 at a point which represents the "extrapolated" boundary (which is extended by the extrapolation length). However, the true density does not vanish outside the boundary. Thus, the diffusion theory does not give a good approximation near the boundary. Rather, the extrapolated boundary conditions are intended to yield the proper density only in the interior of the region of interest several mean free paths away from the surface. Moreover, the diffusion theory together with boundary layers corrrectors are not valid near and in the regions of X for which Σ, f, and g have large variation.

In this paper we shall introduce an alternative model to these classical methods. We shall then establish the mathematical foundations of these new models. Our method consists of using the right physical model in its domain of validity and couple the resulting models. This method involves additional mathematical difficulties related to the matching of equations of these models. However, this approach has several advantages. One of the great advantages is the use of the correct model related to the physical features of the system. We shall then develop the mathematical theory of these new models. This theory is based in a crucial way on the methods we introduced in refs. 14, 15.

The rest of this paper is organized as follows. In the next section we shall introduce and derive our models. In Section 3, we shall establish the existence theory in L^∞ and prove the positivity of the solutions for our models. The results of this section are based on ideas of Papanicolaou,[11] the maximum principle, and the transmission time marching algorithm. In Section 4, we establish the existence theory in L^2. This result is based on

Hille–Yosida theory and the methods of ref. 14. In Section 5, we study the asymptotic behaviour of the solutions to our models. In Section 6, we study the convergence properties of the algorithm resulting from the applications of the transmission time marching algorithm to the approximations of the solutions to our new models. These results are based in a crucial way on the methods we introduced in refs. 14, 15. Finally, we give in Section 7 some concluding remarks and extensions.

2. PROPOSED MODELS

Let X_1, X_2 be two opens of $\mathbb{R}^3$ such that

$$X_1, X_2 \subset X, \qquad \bar{X}_1 \cup \bar{X}_2 = \bar{X}, \qquad X_2 = X - \bar{X}_1, \qquad \Gamma_{12} = \partial X_1 \cap X = \partial X_2 \cap X$$

Assume that X_1 is the domain where the diffusion theory gives a poor approximation to the density of particles while in X_2 the diffusion theory is correct. Then we propose the following physical model consisting of two models: the transport model used in X_1 and the diffusion model used in X_2,

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon} v \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon^2} \Sigma(K - I)\, u_\varepsilon + \gamma u_\varepsilon \text{ in } X_1 \times S^2 \times]0, +\infty \tag{10}$$

$$u_\varepsilon(x, v, t) = 0, \qquad (x, v) \in ((\partial X_1 \setminus \Gamma_{12}) \times S^2)_-, \quad t > 0 \tag{11}$$

$$u_\varepsilon(x, v, t) = u(x, t), \qquad (x, v) \in (\Gamma_{12} \times S^2)_-, \quad t > 0 \tag{12}$$

$$u_\varepsilon(x, v, 0) = u_I(x) \qquad (x, v) \in X_1 \times S^2 \tag{13}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + \gamma u \qquad \text{in} \quad X_2 \times]0, +\infty[\tag{14}$$

$$u(x, t) = 0, \qquad x \in \partial X_2 \setminus \Gamma_{12}, \quad t > 0 \tag{15}$$

$$u \int_{S^2,\, v \cdot n_1 \geqslant 0} v \cdot n_1 \, dx - \frac{\varepsilon}{\Sigma(x)} \int_{S^2,\, v \cdot n_1 \geqslant 0} \frac{\partial u}{\partial x_i} D_i v \cdot n_1 \, dv$$

$$= \int_{S^2,\, v \cdot n_1 \geqslant 0} u_\varepsilon(x, v, t)\, v \cdot n_1 \, dv \qquad x \in \Gamma_{12}, \quad t > 0 \tag{16}$$

$$u(x, 0) = u_I(x), \qquad x \in X_2 \tag{17}$$

where n denotes the unit exterior normal vector to X_1. We shall refer to this model as the model (α). We shall briefly describe the derivation of the matching conditions (12) and (16) between the two models. Set $u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \psi_\varepsilon$. We then make the approximation

$$u_\varepsilon(x, v, t) = u_0(x, t) = u(x, t) \qquad (x, v) \in (\Gamma_{12} \times S^2)_-, \quad t > 0$$

To obtain the transmission condition (16), we assume that the flux of particles

$$J_{\varepsilon, i} = \int_{S^2} v_i u_\varepsilon(x, v, t)\, dv$$

is conserved through Γ_{12}, which is justified from physical point of view.

$$\begin{aligned} J_\varepsilon \cdot n &= \int_{v \in S^2,\, v \cdot n \geqslant 0} u_\varepsilon(x, v, t)\, v \cdot n\, dv + \int_{v \in S^2,\, v \cdot n \leqslant 0} u_\varepsilon(x, v, t)\, v \cdot n\, dv \\ &= (J_\varepsilon \cdot n)^+ + (J_\varepsilon \cdot n)^- \end{aligned}$$

If we make the approximation: $u_\varepsilon = u_0 + \varepsilon u_1$ then

$$u_\varepsilon = u - \frac{\varepsilon D_i}{\Sigma(x)} \frac{\partial u}{\partial x_i}$$

and

$$\begin{aligned} \int_{S^2,\, v \cdot n_1 \geqslant 0} u_\varepsilon(x, v, t)\, v \cdot n_1\, dv &= \int_{S^2,\, v \cdot n_1 \geqslant 0} \left(u - \frac{\varepsilon D_i}{\Sigma(x)} \frac{\partial u}{\partial x_i} \right) v \cdot n_1\, dv \\ &= u \int_{S^2,\, v \cdot n_1 \geqslant 0} v \cdot n_1\, dv \\ &\quad - \frac{\varepsilon}{\Sigma(x)} \int_{S^2,\, v \cdot n_1 \geqslant 0} \frac{\partial u}{\partial x_i} D_i v \cdot n_1\, dv \end{aligned}$$

The transport problem is difficult to solve for small mean free path. Since in this case the diffusion theory is not valid near (but valid outside a neighbourhood of the boundary) the boundary we need only take X_1 to be a small neighborhood of the boundary. Therefore, we need only solve the transport equations on a small domain X_1. The two problems (the diffusion equations and the transport equations) are only coupled by their boundary conditions. Therefore, they can be solved by two independent solution techniques.

Assuming now that the diffusion theory gives a good approximation of the transport equations everywhere except on the surface (for the boundary conditions). Then in this case we take $X_2 = X$. The model we propose corresponds to the following physical model consisting of two models: the transport model used in X_1 and the diffusion model used globally in $X_2 = X$,

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon} v \cdot \nabla u_\varepsilon + \frac{1}{\varepsilon^2} \Sigma (K - I)\, u_\varepsilon + \gamma u_\varepsilon \qquad \text{in} \quad X_1 \times S^2 \times]0, +\infty[\tag{18}$$

$$u_\varepsilon(x, v, t) = 0, \qquad (x, v) \in (\partial X_1 \setminus \Gamma_{12}) \times S^2)_-, \quad t > 0 \tag{19}$$

$$u_\varepsilon(x, v, t) = u(x, t), \qquad (x, v) \in (\Gamma_{12} \times S^2)_-, \quad t > 0 \tag{20}$$

$$u_\varepsilon(x, v, t) = u_I(x) \qquad (x, v) \in X_1 \times S^2 \tag{21}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + \gamma u \qquad \text{in} \quad X \times]0, +\infty[\tag{22}$$

$$u(x, t) = 0, \qquad x \in \partial X \setminus \partial X_1, \quad t > 0 \tag{23}$$

$$(J \cdot n)_- = \int_{S^2,\, v \cdot n_1 \leqslant 0} u_\varepsilon(x, v, t)\, v \cdot n_1\, dv \qquad x \in \partial X_1 \setminus \Gamma_{12}, \quad t > 0 \tag{24}$$

$$u(x, 0) = u_I(x), \qquad x \in X \tag{25}$$

The resulting model will be refered to as the model (β). The derivation of the matching conditions (20) and (24) between the two models is similar to the derivation of (12) and (16) for the coupled models (10)–(17). For the transmission condition (24), we use the kinetic definition of the flux $J \odot n$. In practice this amounts to a discrete kinetic definition of the flux $J \odot n$.

In fact a large variety of boundary conditions are possible. Without changing the global solver (for the diffusion equations) our coupled models give an easy way of supplementing and testing a large variety of transport boundary conditions.

The general algorithm that we propose for the solution of our models (α) and (β) is to integrate the evolution problem (10)–(17) and (18)–(25) with respect to time using the transmission time marching algorithm introduced and studied in refs. 10, 13, 14, 15 (see also the references therein). This integration in time is then achieved by the following uncoupled semi-explicit algorithm, where the operators are treated implicitly inside each subdomain and where one of the coupling boundary conditions is treated explicitly and the other is treated implicitly. For the model (α), we obtain

- Set $u_\varepsilon^{(0)} = u_I$ and $u^{(0)} = u_I$
- then, for $n \geqslant 0$, $u_\varepsilon^{(n)}$ and $u^{(n)}$ being known, solve successively

$$\frac{u_\varepsilon^{(n+1)} - u_\varepsilon^{(n)}}{\Delta t} = -\frac{1}{\varepsilon} v \cdot \nabla u_\varepsilon^{(n+1)} + \frac{1}{\varepsilon^2} \Sigma(K - I)\, u_\varepsilon^{(n+1)} + \gamma u_\varepsilon^{(n+1)} \quad \text{in} \quad X_1 \times S^2 \tag{26}$$

$$u_\varepsilon^{(n+1)}(x, v) = 0, \qquad (x, v) \in ((\partial X_1 \setminus \Gamma_{12}) \times S^2)_- \tag{27}$$

$$u_\varepsilon^{(n+1)}(x, v) = u^{(n+1)}(x), \qquad (x, v) \in (\Gamma_{12} \times S^2)_- \tag{28}$$

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u^{(n+1)}}{\partial x_j} + \gamma u^{(n+1)}, \qquad x \in X_2 \tag{29}$$

$$u^{(n+1)}(x) = 0, \qquad x \in \partial X_2 \setminus \Gamma_{12} \tag{30}$$

$$u^{(n+1)} \int_{S^2,\, v \cdot n_1 \geqslant 0} v \cdot n_1 \, dv - \frac{\varepsilon}{\Sigma(x)} \int_{S^2,\, v \cdot n_1 \geqslant 0} \frac{\partial u^{(n+1)}}{\partial x_i} D_i v \cdot n_1 \, dv$$
$$= \int_{S^2,\, v \cdot n_1 \geqslant 0} u_\varepsilon^{(n)}(x, v, t)\, v \cdot n_1 \, dv \qquad x \in \Gamma_{12} \tag{31}$$

For the model (β), we obtain

- Set $u_\varepsilon^{(0)} = u_I$ and $u^{(0)} = u_I$
- then, for $n \geqslant 0$, $u_\varepsilon^{(n)}$ and $u^{(n)}$ being known, solve successively

$$\frac{u_\varepsilon^{(n+1)} - u_\varepsilon^{(n)}}{\Delta t} = -\frac{1}{\varepsilon} v \cdot \nabla u_\varepsilon^{(n+1)} + \frac{1}{\varepsilon^2} \Sigma(K - I)\, u_\varepsilon^{(n+1)} + \gamma u_\varepsilon^{(n+1)} \quad \text{in} \quad X_1 \times S^2 \tag{32}$$

$$u_\varepsilon^{(n+1)}(x, v) = 0, \qquad (x, v) \in ((\partial X_1 \setminus \Gamma_{12}) \times S^2)_- \tag{33}$$

$$u_\varepsilon^{(n+1)}(x, v) = u^{(n+1)}(x), \qquad (x, v) \in (\Gamma_{12} \times S^2)_- \tag{34}$$

$$\frac{u^{(n+1)} - u^{(n)}}{\Delta t} = \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u^{(n+1)}}{\partial x_j} + \gamma u^{(n+1)}, \qquad x \in X \tag{35}$$

$$u^{(n+1)}(x) = 0, \qquad x \in \partial X \setminus \partial X_1 \tag{36}$$

$$J^{(n+1)} \odot n = \int_{S^2,\, v \cdot n_1 \leqslant 0} u_\varepsilon^{(n)}(x, v, t)\, v \cdot n_1 \, dv \qquad x \in \partial X_2 \cap \partial X_1 \tag{37}$$

The problems (26)–(28) and (29)–(31) respectively (32)–(34) and (35)–(37) are fully uncoupled. Therefore, they can be solved by two independent solvers.

Before we establish the mathematical foundations of these new models, we shall give the integral formulations of the transport equations (12)–(15). We then describe the infinite strip problem for which we shall develop our analysis.

Problem (10)–(13) can be written as follows

$$\frac{\partial u_\varepsilon}{\partial t} + v' \cdot \nabla u_\varepsilon + \Sigma' u_\varepsilon = K' u_\varepsilon \qquad \text{in} \quad X_1 \times S^2 \times]0, +\infty[$$

$$u_\varepsilon(x, v, t) = 0, \qquad (x, v) \in ((\partial X_1 \setminus \Gamma_{12}) \times S^2)_-, \qquad t > 0$$

$$u_\varepsilon(x, v, t) = u(x, t), \qquad (x, v) \in (\Gamma_{12} \times S^2)_-, \quad t > 0$$

$$u_\varepsilon(x, v, 0) = u_I(x) \qquad (x, v) \in X_1 \times S^2$$

where

$$v' = \frac{1}{\varepsilon} v$$

$$\Sigma' = \frac{\Sigma}{\varepsilon^2} - \gamma$$

$$K' = \frac{\Sigma}{\varepsilon^2} K$$

We shall now derive the integral formulation of the scaled transport equations. We proceed as in the case of unscaled transport equations. We consider first the problem with homogeneous boundary conditions. We set

$$g(x, v', t) = K' u_\varepsilon(x, v, t)$$

and then write the problem in the abstract form

$$\frac{du_\varepsilon}{dt} = (A - \Sigma')\, u_\varepsilon + g$$

Using the semigroup G_Σ spanned by the operator $A - \Sigma$, we obtain

$$u_\varepsilon(t) = G_\Sigma(t)\, u_0 + \int_0^t G_\Sigma(t-s)\, g(s)\, ds$$

where

$$G_\Sigma(t)\, u_0 = u_0(x - v't, v) \exp\left(-\int_0^t \Sigma'(x - v's, v)\, ds \right) Y(t(x, v') - t)$$

where Y denotes the Heaviside function ($Y(s) = 0$ if $s < 0$ and $Y(s) = 1$ if $s > 0$).

$$\int_0^t G_\Sigma(t-s)\, g(s) = \int_0^t G_\Sigma(s)\, g(t-s)\, ds$$

$$G_\Sigma(s)\, g(t-s) = g(x - v's, v, t-s) \exp\left(-\int_0^s \Sigma'(x - v'\tau, v)\, d\tau \right) Y(t(x, v') - s)$$

We then have

$$\begin{aligned} u_\varepsilon(t) &= u_{\varepsilon 0}(x - v't, v) \exp\left(-\int_0^t \Sigma'(x - v's, v)\, ds \right) Y(t(x, v') - t) \\ &\quad + \int_0^t g(x - v's, v, t-s) \exp\left(-\int_0^s \Sigma'(x - v'\tau, v)\, d\tau \right) Y(t(x, v') - s)\, ds \end{aligned}$$

For the complete problem (problem with nonhomogeneous boundary conditions) we add the following term

$$Y(t - t(x, v')) \exp\left(-\int_0^t (x, v')\, \tilde{\Sigma}(s)\, ds \right) u(x - v't(x, v'), v, t - t(x, v'))$$

We then obtain the integral formulation for the problem (10)– (13)

$$\begin{aligned} u_\varepsilon(t) &= u_{\varepsilon 0}(x - v't, v) \exp\left(-\int_0^t \Sigma'(x - v's, v \right) Y(t(x, v') - t) \\ &\quad + \int_0^t g(x - v's, v, t-s) \exp\left(-\int_0^s \Sigma'(x - v'\tau, v)\, d\tau \right) Y(t(x, v') - s)\, ds \\ &\quad + Y(t - t(x, v')) \exp\left(-\int_0^{t(x, v')} \Sigma'(x - vs, v)\, ds \right) \\ &\quad \times u(x - v't(x, v'), v, t - t(x, v')) \end{aligned} \tag{38}$$

To develop the mathematical theory of these new models, we shall study the problem of infinite strip with isotropic collision operator. This problem corresponds to solving the transport equations (4) in the open of $\mathbb{R}^3$

$$\{(x, x_2, x_3) \in \mathbb{R}^3;\ -1 \leqslant x_1 \leqslant 1\} \tag{39}$$

with $V = S^2$ the unit sphere of $\mathbb{R}^3$, Σ is constant and positive, and the collision kernel is also constant. It is not difficult to prove that solving the transport equations (4) in the open (39) of $\mathbb{R}^3$ with absorbing boundary conditions and initial condition depending only on x_1, is equivalent to solving the following equations in the interval $]-1, 1[$ of $\mathbb{R}$, μ being the projection of the velocity of neutrons on the axis Ox_1

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon}{\partial x} + \frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon + \gamma u_\varepsilon \qquad \text{in} \quad]-1, 1[\times[-1, 1]\times]0, +\infty[\tag{40}$$

$$u_\varepsilon(-1, \mu, t) = 0, \qquad \mu > 0, \quad t > 0 \tag{41}$$

$$u_\varepsilon(1, \mu, t) = 0, \qquad \mu < 0, \quad t > 0 \tag{42}$$

$$u_\varepsilon(x, \mu, 0) = u_I(x) \qquad (x, \mu) \in]-1, 1[\times[-1, 1] \tag{43}$$

where we have

$$V = [-1, 1], \qquad f \equiv 1, \qquad Kg = \int_{-1}^{1} g(\mu)\frac{d\mu}{2}, \qquad D_i(\mu) = \mu$$

We shall take

$$X =]0, 1[, \qquad X_1 =]0, a[, \qquad X_2 =]a, 1[\qquad (0 < a < 1)$$

Hence we have

$$\begin{aligned} a_{ij} &= \frac{1}{\Sigma(x)}(1, v_i D_j)\, V \\ &= \frac{1}{\Sigma(x)}\int_{-1}^{1} v_i v_j\, dv \\ &= \frac{1}{\Sigma(x)}\int_{-1}^{1} \mu^2 \frac{d\mu}{2} \\ &= \frac{1}{3}\frac{1}{\Sigma(x)} \end{aligned} \tag{44}$$

On the other hand the transmission condition (16) becomes

$$u\int_0^1 \mu \frac{d\mu}{2} - \frac{\varepsilon}{\Sigma}\frac{\partial u}{\partial x}\int_0^1 D(\mu)\,\mu\frac{d\mu}{2} = \int_0^1 u_\varepsilon(a,\mu,t)\,\mu\frac{d\mu}{2}$$

Hence we have

$$\begin{aligned}\frac{1}{4}u - \frac{\varepsilon}{\Sigma}\frac{\partial u}{\partial x_i}\int_0^1 \mu^2\frac{d\mu}{2} &= \frac{1}{4}u - \frac{1}{6}\frac{\varepsilon}{\Sigma}\frac{\partial u}{\partial x_i}\\ &= \frac{1}{2}\int_0^1 u_\varepsilon(a,\mu,t)\,\mu\,d\mu\end{aligned}$$

And then we obtain

$$\frac{1}{2}u - \frac{\varepsilon}{3\Sigma}\frac{\partial u}{\partial x_i} = \int_0^1 u_\varepsilon(a,\mu,t)\,\mu\,d\mu$$

The model (α) then becomes

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon}{\partial x} + \frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon + \gamma u_\varepsilon \qquad \text{in} \quad]0,a[\times[-1,1]\times]0,+\infty[\tag{45}$$

$$u_\varepsilon(0,\mu,t) = 0, \qquad \mu > 0, \quad t > 0 \tag{46}$$

$$u_\varepsilon(a,\mu,t) = u(a,t), \qquad \mu < 0, \quad t > 0 \tag{47}$$

$$u_\varepsilon(x,\mu,0) = u_I(x) \qquad (x,v) \in]0,a[\times[-1,1] \tag{48}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial u}{\partial x}\right) + \gamma u \qquad \text{in} \quad]a,1[\times]0,+\infty[\tag{49}$$

$$u(1,t) = 0, \qquad t > 0 \tag{50}$$

$$\frac{1}{2}u(a,t) - \frac{\varepsilon}{3\Sigma}\frac{\partial u}{\partial x}(a,t) = \int_0^1 u_\varepsilon(a,\mu,t)\,\mu\,d\mu \qquad t > 0 \tag{51}$$

$$u(x,0) = u_I(x), \qquad x \in]a,1[\tag{52}$$

We shall make the assumptions that γ is constant. Our results are valid for more general coefficients Σ and γ. We shall also assume that $\gamma < 0$. This assumption ($\gamma < 0$) is important for the existence of a solution to the stationary (diffusion) problem.

The steady problem corresponding to Eqs. (45)–(52) is as follows

$$-\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon}{\partial x}+\frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon+\gamma u_\varepsilon=0 \qquad \text{in} \quad]0,a[\times[-1,1] \tag{53}$$

$$u_\varepsilon(0,\mu)=0, \qquad \mu>0 \tag{54}$$

$$u_\varepsilon(a,\mu)=u(a), \qquad \mu<0 \tag{55}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial u}{\partial x}\right)+\gamma u=0 \qquad \text{in} \quad]a,1[\tag{56}$$

$$u(1)=0 \tag{57}$$

$$\frac{1}{2}u(a)-\frac{\varepsilon}{3\Sigma}\frac{du}{dx}(a)=\int_0^1 u_\varepsilon(a,\mu)\,\mu\,d\mu \tag{58}$$

3. ANALYSIS IN L^∞: EXISTENCE THEORY AND POSITIVITY

In this section, we shall study the positivity and existence theory of the solution to the problem (45)–(52) in L^∞. For this purpose we introduce an iterative process which is based on the transmission time marching algorithm and ideas of Papanicolaou.[11] More precisely, we have the following result.

Theorem 3.1. Assume that $u_I\in L^\infty(X)$ and $u_I\geqslant 0$ a.e. Then the coupled problem has a unique solution (u_ε,u) in $L^\infty(X_1\times V)\times L^\infty(X_2)$. Moreover, (u_ε,u) satisfies $u_\varepsilon\geqslant 0$ and $u\geqslant 0$ a.e.

Proof. To prove this theorem, we introduce the following iterative process

$$u_\varepsilon^{(0)}(x,v,t)=u_{\varepsilon 0}(x-v't,v)\exp\left(-\int_0^t \Sigma'(x-v's,v)\,ds\right)Y(t(x,v')-t)$$

$(u_\varepsilon^{(n+1)},u^{(n+1)})$ is a solution of the coupled problem

$$\frac{\partial u_\varepsilon^{(n+1)}}{\partial t}=-\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon^{(n+1)}}{\partial x}+\frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon^{(n+1)}+\gamma u_\varepsilon^{(n+1)}$$
$$\text{in} \quad]0,a[\times[-1,1]\times]0,+\infty[\tag{59}$$

$$u_\varepsilon^{(n+1)}(0,\mu,t)=0, \qquad \mu>0, \quad t>0 \tag{60}$$

$$u_\varepsilon^{(n+1)}(a, \mu, t) = u^{(n+1)}(a, t), \qquad \mu < 0, \quad t > 0 \tag{61}$$

$$u_\varepsilon^{(n+1)}(x, \mu, 0) = u_I(x) \qquad (x, v) \in]0, a[\times [-1, 1] \tag{62}$$

$$\frac{\partial u^{(n+1)}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)} \frac{\partial u^{(n+1)}}{\partial x}\right) + \gamma u^{(n+1)} \qquad \text{in} \quad]a, 1[\times]0, +\infty[\tag{63}$$

$$u^{(n+1)}(1, t) = 0, \qquad t > 0 \tag{64}$$

$$\frac{1}{2} u^{(n+1)}(a, t) - \frac{\varepsilon}{3\Sigma} \frac{\partial u^{(n+1)}}{\partial x_i}(a, t) = \int_0^1 u_\varepsilon^{(n)}(a, \mu, t)\, \mu \, d\mu \qquad t > 0 \tag{65}$$

$$u^{(n+1)}(x, 0) = u_I(x), \qquad x \in]a, 1[\tag{66}$$

We first assume $n = 0$. The boundary term (65) can be written as follows

$$u^{(1)} + \beta \frac{\partial u^{(1)}}{\partial n_2} = \psi$$

where $\beta \geqslant 0$ and $\psi \geqslant 0$ (since $u_\varepsilon^{(0)} \geqslant 0$). Moreover, $u^{(1)}(x, 0) \geqslant 0$.

By maximum principle we obtain

$$0 \leqslant u^{(1)} \leqslant \|u_I\|_\infty \tag{67}$$

On the other hand, using (38), the solution of Problem (45)– (48), can be written as follows

$$\begin{aligned} u_\varepsilon^{(1)}(t) = {} & u_{\varepsilon 0}(x - v't, v) \exp\left(-\int_0^t \Sigma'(x - v', v)\, ds\right) Y(t(x, v') - t) \\ & + \int_0^t K' u_\varepsilon^{(1)}(x - v's, v, t - s) \exp\left(-\int_0^s \Sigma'(x - v'\tau, v)\, d\tau\right) \\ & \times Y(t(x, v') - s)\, ds \\ & + Y(t - t(x, v')) \exp\left(-\int_0^{t(x, v')} \Sigma'(x - v's, v)\, ds\right) \\ & \times u^{(1)}(x - v't(x, v'), v, t - t(x, v')) \end{aligned} \tag{68}$$

Using the positivity of the collision integral and $u^{(1)}$, we conclude that

$$u_\varepsilon^{(1)} \geqslant 0$$

By induction we show that

$$u^{(n)} \geqslant 0 \qquad \forall n \geqslant 1$$

$$u_\varepsilon^{(n)} \geqslant 0 \qquad \forall n \geqslant 1$$

Next, we shall show that

$$u^{(n)} \leqslant \|u_I\|_\infty \qquad \forall n \geqslant 1 \tag{69}$$

$$u_\varepsilon^{(n)} \leqslant \|u_I\|_\infty \qquad \forall n \geqslant 1 \tag{70}$$

The boundedness of $u^{(1)}$ is given in (68). Using (68), we obtain

$$u_\varepsilon^{(1)} \leqslant \|u_{\varepsilon 0}\|_\infty$$

The relations (69)–(70) are then obtained by induction.
We shall now prove that

$$u^{(n+1)} \geqslant u^{(n)}$$

$$u_\varepsilon^{(n+1)} \geqslant u_\varepsilon^{(n)}$$

By comparison principle, we prove that

$$u^{(1)} \leqslant u^{(0)}$$

Using again (68), we obtain

$$u_\varepsilon^{(1)} \geqslant u_\varepsilon^{(0)}$$

By induction we show that

$$u^{(n+1)} \geqslant u^{(n)}$$

$$u_\varepsilon^{(n+1)} \geqslant u_\varepsilon^{(n)}$$

The sequences $u^{(n)}$ and $u_\varepsilon^{(n)}$ are nonnegative increasing and bounded. Therefore, they have nonnegative limits u and u_ε. Moreover by monotone convergence theorem the limit (u_ε, u) satisfies the system (45)–(52).

We shall now prove the uniqueness of the solution to the problem (45)–(52). Let $(u_{1\varepsilon}, u_1)$ and $(u_{2\varepsilon}, u_2)$ be a pair of solutions to Problem (45)–(52), then

$$u_\varepsilon = u_{2\varepsilon} - u_{1\varepsilon}$$

$$u = u_2 - u_1$$

satisfy

$$\frac{\partial u_\varepsilon}{\partial t} = -\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon}{\partial x} + \frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon + \gamma u_\varepsilon \qquad \text{in} \quad]0, a[\times[-1, 1]\times]0, +\infty[\tag{71}$$

$$u_\varepsilon(0, \mu, t) = 0, \qquad \mu > 0, \quad t > 0 \tag{72}$$

$$u_\varepsilon(a, \mu, t) = u(a, t), \qquad \mu < 0, \quad t > 0 \tag{73}$$

$$u_\varepsilon(x, \mu, 0) = 0 \qquad (x, v) \in]0, a[\times[-1, 1] \tag{74}$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial u}{\partial x}\right) + \gamma u \qquad \text{in} \quad]a, 1[\times]0, +\infty[\tag{75}$$

$$u(1, t) = 0, \qquad t > 0 \tag{76}$$

$$\frac{1}{2}u(a, t) - \frac{\varepsilon}{3\Sigma}\frac{\partial u}{\partial x_i}(a, t) = \int_0^1 u_\varepsilon(a, \mu, t)\,\mu\,d\mu \qquad t > 0 \tag{77}$$

$$u(x, 0) = 0, \qquad x \in]a, 1[\tag{78}$$

By maximum principle we show that the solution u of Problem (77)–(78) is identically 0. It is then clear that u_ε is identically 0. And the theorem is proved. ▮

4. EXISTENCE THEORY IN L^2

In this section we shall give an existence result for the solution to our models in L^2. This result is based on Hille–Yosida theory and the methods we introduced in refs. 14, 15. We shall work in the Hilbert space

$$H = L^2(X_1 \times V) \times L^2(X_2)$$

with the following norm

$$\|(w_1, w_2)\| = (\|w_1\|^2_{L^2(X_1\times V)} + \|w_2\|^2_{L^2(X_2)})^{1/2}$$

We have the following existence and uniqueness result.

Theorem 4.1. Assume that $u_I \in L^2(X)$. Then the coupled problem (45)–(52) has a unique strong solution.

Proof. Let A be the operator defined in H by

$$A(w_1, w_2) = \begin{pmatrix} \dfrac{\mu}{\varepsilon}\dfrac{\partial w_1}{\partial x} \\ -\dfrac{1}{3\Sigma} w_2'' \end{pmatrix} \tag{79}$$

$$D(A) = \left\{ \begin{array}{ll} (w_1, w_2) \in H \mid \mu \dfrac{\partial w_1}{\partial x} \in L^2(X_1 \times V), & \text{and} \quad w_2'' \in L^2(X_2) \\ w_2(1) = 0, \qquad w_1(0, \mu) = 0 \quad \mu > 0 & \\ w_1(a, \mu) = w_2(a) \qquad \mu < 0 & \\ \dfrac{1}{2} w_2(a) - \dfrac{1}{3}\dfrac{\varepsilon}{\Sigma} w_2'(a) = \int_0^1 \mu w_1(a, \mu)\, d\mu & \end{array} \right\}$$

Let B be the operator defined on H by

$$B(w_1, w_2) = \begin{pmatrix} -\dfrac{1}{\varepsilon^2}\Sigma(K - I)\, w_1 - \gamma w_1 \\ -\gamma w_2 \end{pmatrix} \tag{80}$$

B is linear continu in H. By a perturbation result if we prove that A is the infinitesimal generator of a $\mathscr{C}^0$ semigroup then $A + B$ is also an infinitesimal generator of $\mathscr{C}^0$ semigroup and Theorem 4.1 will be proved. It is clear that $D(A)$ is dense in H. We shall use the theorem of Hille–Yosida.[6]

Let λ be a real number and let $f \in H$. We shall study the problem

$$\text{Find } w \in D(A) \text{ solution of} \tag{81}$$

$$Aw + \lambda w = f \tag{82}$$

which corresponds to finding $w \in D(A)$ such that

$$\frac{\mu}{\varepsilon}\frac{\partial w_1}{\partial x} + \lambda w_1 = f_1 \tag{83}$$

$$-\frac{1}{3\Sigma} w''_2 + \lambda w_2 = f_2 \tag{84}$$

By a density argument we may assume that f_1 and f_2 are continuous. By elementary methods we obtain the general solution of Eqs. (83) and (84). Using the boundary conditions we found a unique (explicit) solution to the system (83)–(84).

We shall now prove the estimate

$$\|w\| \leqslant \frac{C}{\lambda - 1} \|f\| \qquad \forall \lambda > 1$$

We first set $\alpha = \frac{1}{3\Sigma}$. Multiplying Eqs. (83)–(84) respectively by $\varphi_1 w_1$, $\varphi_2 w_1$, and $\varphi_3 w_2$, with φ_1, φ_2, and φ_3 positive functions to be precised later, integrating by parts, we obtain

$$\int_a^1 \left(\lambda\varphi_3 - \frac{\alpha}{2}\varphi_3''\right) w_2^2\, dx + \alpha \int_a^1 \varphi_3 \left(\frac{\partial w_2}{\partial x}\right)^2 + \frac{\alpha}{2}\,[\varphi_3' w_2^2]_a^1 - \alpha \left[\varphi_3 w_2 \frac{\partial w_2}{\partial x}\right]_a^1$$

$$= \int_a^1 \varphi_3 w_2 f_2 \tag{85}$$

$$\int_{-1}^0 \int_0^a \left(\lambda\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right) w_1^2 + \int_{-1}^0 \frac{\mu}{2\varepsilon} w_1^2(a,\mu)\,\varphi_1(a) - \int_{-1}^0 \frac{\mu}{2\varepsilon} w_1^2(0,\mu)\,\varphi_1(0)$$

$$= \int_{-1}^0 \int_0^a \varphi_1 f_1 w_1 \tag{86}$$

Similarly, we obtain

$$\int_0^1 \int_0^a \left(\lambda\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right) w_1^2 + \int_0^1 \frac{\mu}{2\varepsilon} w_1^2(a,\mu)\,\varphi_2(a) - \int_0^1 \frac{\mu}{2\varepsilon} w_1^2(0,\mu)\,\varphi_2(0)$$

$$= \int_0^1 \int_0^a \varphi_2 f_1 w_1 \tag{87}$$

Adding the boundary terms in (85), (86), and (87) and using the coupling boundary conditions, we obtain

$$\begin{aligned}
&\int_{-1}^{0}\left(\frac{\mu}{2\varepsilon}w_1^2(a,\mu)\,\varphi_1(a)-\frac{\mu}{2\varepsilon}w_1^2(0,\mu)\,\varphi_1(0)\right)\\
&\quad+\int_0^1\left(\frac{\mu}{2\varepsilon}w_1^2(a,\mu)\,\varphi_2(a)-\frac{\mu}{2\varepsilon}w_1^2(0,\mu)\,\varphi_2(0)\right)\\
&\quad+\frac{\alpha}{2}\,[\varphi_3'w_2^2]_a^1-\alpha\left[\varphi_3 w_2\frac{\partial w_2}{\partial x}\right]_a^1\\
&\quad=w_2^2(a)\int_{-1}^{0}\frac{\mu}{2\varepsilon}\,\varphi_1(a)-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu\varphi_1(0)\,w_1^2(0,\mu)\,d\mu\\
&\quad+\frac{1}{2\varepsilon}\int_0^1\mu w_1^2(a,\mu)\,\varphi_2(a)\,d\mu\\
&\quad-\frac{\alpha}{2}\,\varphi_3'(a)\,w_2^2(a)+\alpha\varphi_3(a)\,w_2(a)\,w_2'(a)\\
&\quad=BC
\end{aligned} \tag{88}$$

We shall now give a lower bound of the boundary terms BC. Using the boundary conditions, we obtain

$$w_2(a)\,w_2'(a)=\frac{3\Sigma}{2\varepsilon}\,w_2^2(a)-\frac{3\Sigma}{\varepsilon}\left(\int_0^1\mu w_1(a,\mu)\,d\mu\right)w_2(a) \tag{89}$$

Using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\frac{3\Sigma}{\varepsilon}\left|w_2(a)\int_0^1\mu w_1(a,\mu)\,d\mu\right|&\leqslant\frac{3\Sigma}{2\varepsilon}\,w_2^2(a)+\frac{3\Sigma}{2\varepsilon}\left(\int_0^1\mu w_1(a,\mu)\,d\mu\right)^2\\
&\leqslant\frac{3\Sigma}{2\varepsilon}\,w_2^2(a)+\frac{3\Sigma}{2\varepsilon}\left(\int_0^1\mu^2 w_1^2(a,\mu)\,d\mu\right)
\end{aligned} \tag{90}$$

Using Eq. (83), we obtain

$$\int_0^1 \mu^2 w_1^2(a,\mu)\, d\mu = 2 \int_0^1 \mu^2 \int_0^a \frac{\partial w_1}{\partial x} w_1$$
$$= 2\varepsilon \int_0^1 \int_0^a \mu w_1 (f_1 - \lambda w_1) \tag{91}$$

Hence we have

$$\int_0^1 \mu^2 w_1^2(a,\mu)\, d\mu = 2\varepsilon \int_0^1 \int_0^a \mu w_1 f_1 - 2\varepsilon \int_0^1 \int_0^a \mu\lambda w_1^2$$
$$\leqslant \varepsilon\lambda \int_0^1 \int_0^a \mu^2 w_1^2 + \frac{\varepsilon}{\lambda} \int_0^1 \int_0^a f_1^2 - 2\varepsilon\lambda \int_0^1 \int_0^a \mu w_1^2 \tag{92}$$

Combining (89), (90), and (92), we obtain

$$\alpha\varphi_3(a)\, w_2(a)\, w_2'(a) \geqslant \varphi_3(a)\, \lambda \int_0^1 \int_0^a (2\mu - \mu^2)\, w_1^2 - \frac{1}{\lambda} \varphi_3(a) \int_0^1 \int_0^a f_1^2 \tag{93}$$

Plugging this in (88), we obtain

$$BC \geqslant \left(\int_{-1}^0 \frac{\mu}{2\varepsilon} \varphi_1(a) - \frac{\alpha}{2} \varphi_3'(a) \right) w_2^2(a)$$
$$+ \varphi_3(a)\, \lambda \int_0^1 \int_0^a (2\mu - \mu^2)\, w_1^2 - \frac{1}{\lambda} \varphi_3(a) \int_0^1 \int_0^a f_1^2$$
$$- \frac{1}{2\varepsilon} \int_{-1}^0 \mu w_1^2(0,\mu)\, \varphi_1(0)\, d\mu + \frac{1}{2\varepsilon} \int_0^1 \mu w_1^2(a,\mu)\, \varphi_2(a)\, d\mu$$
$$\geqslant \left(\int_{-1}^0 \frac{\mu}{2\varepsilon} \varphi_1(a) - \frac{\alpha}{2} \varphi_3'(a) \right) w_2^2(a)$$
$$+ \varphi_3(a)\, \lambda \int_0^1 \int_0^a (2\mu - \mu^2)\, w_1^2 - \frac{1}{\lambda} \varphi_3(a) \int_0^1 \int_0^a f_1^2$$
$$- \frac{1}{2\varepsilon} \int_{-1}^0 \mu w_1^2(0,\mu)\, \varphi_1(0)\, d\mu + \frac{1}{2\varepsilon} \int_0^1 \mu w_1^2(a,\mu)\, \varphi_2(a)\, d\mu \tag{94}$$

Combining (85), (86), (87), and (94), we obtain

$$
\begin{aligned}
&\int_{-1}^{0}\int_{0}^{a}\left(\lambda\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)w_1^2+\int_{-1}^{0}\frac{\mu}{2\varepsilon}w_1^2(a,\mu)\,\varphi_1(a)-\int_{-1}^{0}\frac{\mu}{2\varepsilon}w_1^2(0,\mu)\,\varphi_1(0)\\
&\quad+\int_{0}^{1}\int_{0}^{a}\left(\lambda\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)w_1^2+\int_{0}^{1}\frac{\mu}{2\varepsilon}w_1^2(a,\mu)\,\varphi_2(a)-\int_{0}^{1}\frac{\mu}{2\varepsilon}w_1^2(0,\mu)\,\varphi_2(0)\\
&\quad+\int_{a}^{1}\left(\lambda\varphi_3-\frac{\alpha}{2}\varphi_3''\right)w_2^2\,dx+\alpha\int_{a}^{1}\varphi_3\left(\frac{\partial w_2}{\partial x}\right)^2+\frac{\alpha}{2}\left[\varphi_3'w_2^2\right]_a^1-\alpha\left[\varphi_3 w_2\frac{\partial w_2}{\partial x}\right]_a^1\\
&=\int_{-1}^{0}\int_{0}^{a}\varphi_1 f_1 w_1+\int_{0}^{1}\int_{0}^{a}\varphi_2 f_1 w_1+\int_{a}^{1}\varphi_3 w_2 f_2 \qquad (95)
\end{aligned}
$$

Using Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&\int_{-1}^{0}\int_{0}^{a}\left(\left(\lambda-\frac{1}{2}\right)\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)w_1^2\,dx\,d\mu+\int_{0}^{1}\int_{0}^{a}\left(\left(\lambda-\frac{1}{2}\right)\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)w_1^2\,dx\,d\mu\\
&\quad+\int_{a}^{1}\left(\left(\lambda-\frac{1}{2}\right)\varphi_3-\frac{\alpha}{2}\varphi_3''\right)w_2^2\,dx+\alpha\int_{a}^{1}\varphi_3\left(\frac{\partial w_2}{\partial x}\right)^2dx+BC\\
&\leqslant\frac{1}{2}\int_{-1}^{0}\int_{0}^{a}\varphi_1 f_1^2+\frac{1}{2}\int_{0}^{1}\int_{0}^{a}\varphi_2 f_1^2+\frac{1}{2}\int_{a}^{1}\varphi_3 f_2^2 \qquad (96)
\end{aligned}
$$

Using the lower bound of BC, we obtain

$$
\begin{aligned}
&\int_{-1}^{0}\int_{0}^{a}\left(\left(\lambda-\frac{1}{2}\right)\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)w_1^2\,dx\,d\mu+\int_{0}^{1}\int_{0}^{a}\left(\left(\lambda-\frac{1}{2}\right)\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)w_1^2\,dx\,d\mu\\
&\quad+\int_{a}^{1}\left(\left(\lambda-\frac{1}{2}\right)\varphi_3-\frac{\alpha}{2}\varphi_3''\right)w_2^2\,dx+\alpha\int_{a}^{1}\varphi_3\left(\frac{\partial w_2}{\partial x}\right)^2dx\\
&\quad+\left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\right)w_2^2(a)+\lambda\varphi_3(a)\int_{0}^{1}\int_{0}^{a}(2\mu-\mu^2)\,w_1^2\\
&\quad-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu w_1^2(0,\mu)\,\varphi_1(0)\,d\mu+\frac{1}{2\varepsilon}\int_{0}^{1}\mu w_1^2(a,\mu)\,\varphi_2(a)\,d\mu\\
&\leqslant\frac{1}{2}\int_{-1}^{0}\int_{0}^{a}\varphi_1 f_1^2+\frac{1}{2}\int_{0}^{1}\int_{0}^{a}(2\varphi_3(a)+\varphi_2)\,f_1^2+\frac{1}{2}\int_{a}^{1}\varphi_3 f_2^2 \qquad (97)
\end{aligned}
$$

Consider the functions φ_1 and φ_3 defined as follows:

$$\varphi_1 = \gamma_1 x + \gamma_1, \quad \gamma_1 > 0, \quad x \in [0, a]$$
$$\varphi_2 = -\gamma_2 x + 2\gamma_2, \quad \gamma_2 > 0, \quad x \in [0, a]$$
$$\varphi_3 = \varphi_3(a) + \varphi_3'(a)(x - a)$$

it is now clear that we can choose γ_1 and γ_2 and $\varphi_3(a)$ and $\varphi_3'(a)$ such that φ_1, φ_3, and φ_2 are positives bounded below and above by positive constants independent on λ and such that we have

$$\begin{aligned} \frac{1}{2}\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial \varphi_1}{\partial x} &> \frac{\gamma_1}{2} > 0 \\ \frac{1}{2}\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial \varphi_2}{\partial x} &> \frac{\gamma_2}{2} > 0 \\ \frac{1}{2}\varphi_3 - \frac{\alpha}{2}\varphi_3'' &> \frac{1}{2}\varphi_3(a) > 0 \\ -\frac{1}{4\varepsilon}\varphi_1(a) - \frac{\alpha}{2}\varphi_3'(a) &\geqslant 0 \end{aligned} \tag{98}$$

Using (97) and the above construction of the functions φ_1, φ_2, and φ_3, we obtain

$$\int_{-1}^{0}\int_{0}^{a} (\lambda - 1)\, \varphi_1 w_1^2 \, dx\, d\mu + \int_{0}^{1}\int_{0}^{a} (\lambda - 1)\, \varphi_2 w_1^2 \, dx\, d\mu + \int_{a}^{1} (\lambda - 1)\, \varphi_3 w_2^2 \, dx$$
$$\leqslant \tfrac{1}{2}\int_{-1}^{0}\int_{0}^{a} \varphi_1 f_1^2 + \tfrac{1}{2}\int_{0}^{1}\int_{0}^{a} (2\varphi_3(a) + \varphi_2)\, f_1^2 + \tfrac{1}{2}\int_{a}^{1} \varphi_3 f_2^2$$

Because of our special construction of the function φ_1, φ_3, and φ_2, we conclude that

$$\|w\| \leqslant \frac{C}{\lambda - 1}\|f\| \qquad \forall \lambda > 1$$

And the theorem is proved. ∎

5. ASYMPTOTIC ANALYSIS

In this section, we study the asymptotic behaviour of the solution to Problem (45)–(52). Let $u_{\varepsilon s}$ and u_s denote the solution to the stationary problem corresponding to System (45)–(52), then we have

$$-\frac{1}{\varepsilon}\mu\frac{\partial u_{\varepsilon s}}{\partial x}+\frac{1}{\varepsilon^2}\Sigma(K-I)\,u_{\varepsilon s}+\gamma u_{\varepsilon s}=0 \qquad \text{in} \quad]0,a[\times[-1,1] \tag{99}$$

$$u_{\varepsilon s}(0,\mu)=0, \qquad \mu>0 \tag{100}$$

$$u_{\varepsilon s}(a,\mu)=u_s(a), \qquad \mu<0 \tag{101}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial u_s}{\partial x}\right)+\gamma u_s=0 \qquad \text{in} \quad]a,1[\tag{102}$$

$$u_s(1)=0 \tag{103}$$

$$\frac{1}{2}u_s(a)-\frac{\varepsilon}{3\Sigma}\frac{\partial u_s}{\partial x}(a)=\int_0^1 u_{\varepsilon s}(a,\mu)\,\mu\,d\mu \tag{104}$$

Let $\bar{u}_\varepsilon=u_\varepsilon-u_{\varepsilon s}$ and $\bar{u}=u-u_s$, we then have

$$\frac{\partial \bar{u}_\varepsilon}{\partial t}=-\frac{1}{\varepsilon}\mu\frac{\partial \bar{u}_\varepsilon}{\partial x}+\frac{1}{\varepsilon^2}\Sigma(K-I)\,\bar{u}_\varepsilon+\gamma\bar{u}_\varepsilon \qquad \text{in} \quad]0,a[\times[-1,1]\times]0,+\infty[\tag{105}$$

$$\bar{u}_\varepsilon(0,\mu,t)=0, \qquad \mu>0, \quad t>0 \tag{106}$$

$$\bar{u}_\varepsilon(a,\mu,t)=\bar{u}(a,t), \qquad \mu<0, \quad t>0 \tag{107}$$

$$\bar{u}_\varepsilon(x,\mu,0)=\bar{u}_I(x) \qquad (x,v)\in\,]0,a[\times[-1,1] \tag{108}$$

$$\frac{\partial \bar{u}}{\partial t}=\frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial \bar{u}}{\partial x}\right)+\gamma\bar{u} \qquad \text{in} \quad]a,1[\times]0,+\infty[\tag{109}$$

$$\bar{u}(1,t)=0, \qquad t>0 \tag{110}$$

$$\frac{1}{2}\bar{u}(a,t)-\frac{\varepsilon}{3\Sigma}\frac{\partial \bar{u}}{\partial x}(a,t)=\int_0^1 \bar{u}_\varepsilon(a,\mu,t)\,\mu\,d\mu \qquad t>0 \tag{111}$$

$$\bar{u}(x,0)=\bar{u}_I(x), \qquad x\in\,]a,1[\tag{112}$$

In what follows we shall omit the bar sign. We have the following asymptotic result.

Theorem 5.1. Assume that $u_I \in L^2(X)$, then the solution of Problem (45)–(52) converges in $L^2(X_1 \times V) \times H^1(X_2)$ to the solution of the steady problem (99)–(104) as t tends to ∞.

Proof. We first set $\alpha = \frac{1}{3\Sigma}$. Multiplying Eqs. (105) and (109) respectively by $\varphi_1 u_\varepsilon$, $\varphi_2 u_\varepsilon$ and $\varphi_3 u$ with φ_1, φ_2, and φ_3 positive functions to be precised later, integrating by parts, we obtain

$$\int_0^a \frac{\partial u_\varepsilon}{\partial t} \varphi_1 u_\varepsilon + \frac{\mu}{\varepsilon} \int_0^a \frac{\partial u_\varepsilon}{\partial x} \varphi_1 u_\varepsilon \, dx - \frac{1}{\varepsilon^2} \Sigma \int_0^a (K u_\varepsilon - u_\varepsilon) \, \varphi_1 u_\varepsilon - \gamma \int_0^a u_\varepsilon \varphi_1 u_\varepsilon \, dx = 0$$

$$\int_a^1 \frac{\partial u}{\partial t} \varphi_3 u - \alpha \int_a^1 u'' \varphi_3 u \, dx - \gamma \int_a^1 \varphi_3 u^2 = 0$$

Using Green's formula and integrating with respect to μ on $]-1, 0[$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{-1}^0 \int_0^a \varphi_1 u_\varepsilon^2 + \int_{-1}^0 \int_0^a \left(\left(\frac{1}{\varepsilon^2} \Sigma - \gamma \right) \varphi_1 - \frac{\mu}{2\varepsilon} \frac{\partial \varphi_1}{\partial x} \right) u_\varepsilon^2 \, dx \, d\mu$$
$$+ \int_{-1}^0 \frac{\mu}{2\varepsilon} u_\varepsilon^2(a, \mu) \, \varphi_1(a) - \int_{-1}^0 \frac{\mu}{2\varepsilon} u_\varepsilon^2(0, \mu) \, \varphi_1(0) - \frac{1}{\varepsilon^2} \Sigma \int_0^a K u_\varepsilon \varphi_1 u_\varepsilon = 0 \tag{113}$$

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \varphi_3 u^2 + \alpha \int_a^1 \varphi_3 (u')^2 \, dx - \int_a^1 \left(\gamma \varphi_3 + \frac{\alpha}{2} \varphi_3'' \right) u^2$$
$$+ \frac{\alpha}{2} [u^2 \varphi_3']_a^1 - \alpha \left[\varphi_3 u \frac{\partial u}{\partial x} \right]_a^1 = 0 \tag{114}$$

Similarly we obtain

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^a \varphi_2 u_\varepsilon^2 + \int_0^1 \int_0^a \left(\left(\frac{1}{\varepsilon^2} \Sigma - \gamma \right) \varphi_2 - \frac{\mu}{2\varepsilon} \frac{\partial \varphi_2}{\partial x} \right) u_\varepsilon^2 \, dx \, d\mu$$
$$+ \int_0^1 \frac{\mu}{2\varepsilon} u_\varepsilon^2(a, \mu) \, \varphi_2(a) - \int_0^1 \frac{\mu}{2\varepsilon} u_\varepsilon^2(0, \mu) \, \varphi_2(0) - \frac{1}{\varepsilon^2} \Sigma \int_0^1 \int_0^a K u_\varepsilon \varphi_2 u_\varepsilon = 0 \tag{115}$$

Adding the boundary terms in (113), (114), and (115) and using the boundary conditions, we obtain

$$\int_{-1}^{0}\left(\frac{\mu}{2\varepsilon}u_\varepsilon^2(a,\mu)\,\varphi_1(a)-\frac{\mu}{2\varepsilon}u_\varepsilon^2(0,\mu)\,\varphi_1(0)\right)$$

$$+\int_{0}^{1}\left(\frac{\mu}{2\varepsilon}u_\varepsilon^2(a,\mu)\,\varphi_2(a)-\frac{\mu}{2\varepsilon}u_\varepsilon^2(0,\mu)\,\varphi_2(0)\right)+\frac{\alpha}{2}\,[\varphi_3'u^2]_a^1-\alpha\left[\varphi_3 u\frac{\partial u}{\partial x}\right]_a^1$$

$$=u^2(a)\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu\varphi_1(0)\,u_\varepsilon^2(0,\mu)\,d\mu$$

$$+\frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu-\frac{\alpha}{2}\varphi_3'(a)\,u^2(a)+\alpha\varphi_3(a)\,u(a)\,u'(a)$$

$$=BC \tag{116}$$

Proceeding as in the previous section, we obtain the following lower bound for the boundary terms

$$BC\geqslant\left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\right)u^2(a)-\frac{1}{\varepsilon}\varphi_3(a)\int_{0}^{1}\mu^2u_\varepsilon^2(a,\mu)$$
$$-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu+\frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu \tag{117}$$

Combining (113), (114), and (115), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{0}\int_{0}^{a}\varphi_1u_\varepsilon^2+\int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma-\gamma\right)\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu$$

$$-\frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\int_{0}^{a}Ku_\varepsilon\varphi_1u_\varepsilon$$

$$+\frac{1}{2}\frac{d}{dt}\int_{0}^{1}\int_{0}^{a}\varphi_2u_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma-\gamma\right)\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu$$

$$-\frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\int_{0}^{a}Ku_\varepsilon\varphi_2u_\varepsilon$$

$$+\frac{1}{2}\frac{d}{dt}\int_{a}^{1}\varphi_3u^2+\alpha\int_{a}^{1}\varphi_3(u')^2\,dx-\int_{a}^{1}\left(\gamma\varphi_3+\frac{\alpha}{2}\varphi_3''\right)u^2+BC=0 \tag{118}$$

Next we shall give an estimate of the collision integral terms. We have using Cauchy–Schwarz inequality

$$\left|\int_{-1}^{0}\int_{0}^{a}(Ku_\varepsilon)\,\varphi_1 u_\varepsilon\right| \leqslant \tfrac{1}{2}\int_{-1}^{0}\int_{0}^{a}\varphi_1 (Ku_\varepsilon)^2(x,\mu)\,d\mu\,dx$$

$$+\tfrac{1}{2}\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_\varepsilon^2(x,\mu)\,d\mu\,dx$$

$$\leqslant \int_{-1}^{1}\int_{0}^{a}\left(\int_{-1}^{0}\varphi_1\,d\mu\right)u_\varepsilon^2(x,\mu)\,d\mu\,dx$$

$$+\tfrac{1}{2}\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_\varepsilon^2(x,\mu)\,d\mu\,dx$$

$$\left|\int_{0}^{1}\int_{0}^{a}(Ku_\varepsilon)\,\varphi_2 u_\varepsilon\right| \leqslant \int_{0}^{a}\left(\int_{0}^{1}\varphi_2\,d\mu\right)\left(\int_{-1}^{1}u_\varepsilon^2(x,\mu')\,d\mu'\right)dx$$

$$+\tfrac{1}{2}\int_{0}^{1}\int_{0}^{a}\varphi_2 u_\varepsilon^2(x,\mu)\,d\mu\,dx$$

$$\leqslant \int_{-1}^{1}\int_{0}^{a}\left(\int_{0}^{1}\varphi_2\,d\mu\right)u_\varepsilon^2(x,\mu)\,d\mu\,dx$$

$$+\tfrac{1}{2}\int_{0}^{1}\int_{0}^{a}\varphi_2 u_\varepsilon^2(x,\mu)\,d\mu\,dx$$

We then obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_\varepsilon^2+\int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma-\gamma\right)\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}-\frac{1}{\varepsilon^2}\Sigma\left[\int_{-1}^{0}\varphi_1\,d\mu'+\frac{1}{2}\varphi_1\right]\right.$$
$$\left.-\frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\varphi_2\,d\mu'\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\frac{1}{2}\frac{d}{dt}\int_{0}^{1}\int_{0}^{a}\varphi_2 u_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma-\gamma\right)\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}-\frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu'\right.$$
$$\left.-\frac{1}{\varepsilon^2}\Sigma\left[\int_{0}^{1}\varphi_2\,d\mu'+\frac{1}{2}\varphi_2\right]\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\frac{1}{2}\frac{d}{dt}\int_{a}^{1}\varphi_3 u^2+\alpha\int_{a}^{1}\varphi_3(u')^2\,dx-\int_{a}^{1}\left(\gamma\varphi_3+\frac{\alpha}{2}\varphi_3''\right)u^2+BC\leqslant 0$$

Using (117), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_\varepsilon^2+\int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{2\varepsilon^2}\Sigma-\gamma\right)\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}-\frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu'\right.$$
$$\left.-\frac{1}{\varepsilon^2}\Sigma\int_0^1\varphi_2\,d\mu'\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\frac{1}{2}\frac{d}{dt}\int_{0}^{1}\int_{0}^{a}\varphi_2 u_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{2\varepsilon^2}\Sigma-\gamma\right)\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}-\frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu'\right.$$
$$\left.-\frac{1}{\varepsilon^2}\Sigma\int_0^1\varphi_2\,d\mu'\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\frac{1}{2}\frac{d}{dt}\int_a^1\varphi_3 u^2+\int_a^1\alpha\varphi_3(u')^2\,dx-\int_a^1\left(\gamma\varphi_3+\frac{\alpha}{2}\varphi_3''\right)u^2$$
$$+\left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\right)u^2(a)-\frac{1}{\varepsilon}\varphi_3(a)\int_0^1\mu^2u_\varepsilon^2(a,\mu)$$
$$-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu+\frac{1}{2\varepsilon}\int_0^1\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu\leqslant 0$$

If we choose φ_1 and φ_2 such that

$$\int_{-1}^{0}\varphi_1(x,\mu)\,d\mu=1$$
$$\int_{0}^{1}\varphi_2(x,\mu)\,d\mu=1$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_\varepsilon^2+\int_{-1}^{0}\int_{0}^{a}\left(\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_1-2\right)-\gamma\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\frac{1}{2}\frac{d}{dt}\int_{0}^{1}\int_{0}^{a}\varphi_2 u_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}\left(\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_2-2\right)-\gamma\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\frac{1}{2}\frac{d}{dt}\int_a^1\varphi_3 u^2+\int_a^1\alpha\varphi_3(u')^2\,dx-\int_a^1\left(\gamma\varphi_3+\frac{\alpha}{2}\varphi_3''\right)u^2$$
$$+\left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\right)u^2(a)+\frac{1}{\varepsilon}\int_0^1\mu\left(\frac{1}{2}\varphi_2(a)-\mu\varphi_3(a)\right)u_\varepsilon^2(a,\mu)$$
$$-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu\leqslant 0 \qquad (119)$$

Therefore, it becomes clear that we can choose φ_1, φ_2, and φ_3 positive functions bounded below and above by positive constants independent on μ and x, and satisfying

$$\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_1-2\right)-\gamma\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}=\varphi_1$$

$$\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_2-2\right)-\gamma\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}=\varphi_2$$

$$-\gamma\varphi_3-\frac{\alpha}{2}\varphi_3''=\varphi_3$$

$$\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\geqslant 0$$

$$\frac{1}{2}\varphi_2(a)-\mu\varphi_3(a)\geqslant 0$$

Using (119) and the above construction of the functions φ_1, φ_2 and φ_3, we obtain

$$\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}\varphi_2 u_\varepsilon^2+\int_{a}^{1}\varphi_3 u^2$$
$$\leqslant e^{-2t}\left[\int_{-1}^{0}\int_{0}^{a}\varphi_1 u_{\varepsilon 0}^2+\int_{0}^{1}\int_{0}^{a}\varphi_2 u_{\varepsilon 0}^2+\int_{a}^{1}\varphi_3 u_0^2\right]$$

Because of our special construction of φ_1, φ_2, and φ_3, we obtain

$$\int_{-1}^{0}\int_{0}^{a}u_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}u_\varepsilon^2+\int_{a}^{1}u^2+\int_{a}^{1}(u')^2$$
$$\leqslant Ce^{-2t}\left[\int_{-1}^{0}\int_{0}^{a}u_{\varepsilon 0}^2+\int_{0}^{1}\int_{0}^{a}u_{\varepsilon 0}^2+\int_{a}^{1}u_0^2\right]$$

This concludes the proof of our theorem. ∎

6. CONVERGENCE ANALYSIS OF THE TRANSMISSION TIME MARCHING ALGORITHM

In this section we study the convergence properties of the transmission time marching algorithm applied to the coupled system (45)–(52). Our analysis is based on the methods of refs. 14, 15.

If we apply the transmission time marching algorithm to the model (α) (45)–(52), we obtain the following algorithm

$$\frac{u_\varepsilon^{(n+1)}-u_\varepsilon^{(n)}}{\Delta t}=-\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon^{(n+1)}}{\partial x}+\frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon^{(n+1)}+\gamma u_\varepsilon^{(n+1)}$$

$$\text{in}\quad]0,a[\times[-1,1] \tag{120}$$

$$u_\varepsilon^{(n+1)}(0,\mu)=0,\qquad \mu>0 \tag{121}$$

$$u_\varepsilon^{(n+1)}(a,\mu)=u^{(n+1)}(a),\qquad \mu<0 \tag{122}$$

$$\frac{u^{(n+1)}-u^{(n)}}{\Delta t}=\frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial u^{(n+1)}}{\partial x}\right)+\gamma u^{(n+1)}\qquad \text{in}\quad]a,1[\tag{123}$$

$$u^{(n+1)}(1)=0 \tag{124}$$

$$\frac{1}{2}u^{(n+1)}(a)-\frac{\varepsilon}{3\Sigma}\frac{\partial u^{(n+1)}}{\partial x_i}(a)=\int_0^1 u_\varepsilon^{(n)}(a,\mu)\,\mu\,d\mu \tag{125}$$

$$u_\varepsilon^{(0)}(x,\mu)=u_I(x),\qquad (x,\mu)\in\,]0,a[\times[-1,1],\quad u^{(0)}(x)=u_I(x),$$

$$x\in\,]a,1[\tag{126}$$

We have the following convergence theorem.

Theorem 6.1. Assume that $u_I\in L^2(X)$. Then the solution to the coupled problem (45)–(52) converges in $L^2(X_1\times V)\times H^1(X_2)$ as n tends to ∞ to the solution of the corresponding steady coupled problem (99)– (104).

Proof. Setting $c=\frac{1}{\Delta t}$, $u_\varepsilon=u_\varepsilon^{(n+1)}-u_{\varepsilon s}$, $f_\varepsilon=u_\varepsilon^{(n)}-u_{\varepsilon s}$, $u=u^{(n+1)}-u_s$, $f=u^{(n)}-u_s$, we obtain

$$c(u_\varepsilon-f_\varepsilon)=-\frac{1}{\varepsilon}\mu\frac{\partial u_\varepsilon}{\partial x}+\frac{1}{\varepsilon^2}\Sigma(K-I)\,u_\varepsilon+\gamma u_\varepsilon\qquad \text{in}\quad]0,a[\times[-1,1] \tag{127}$$

$$u_\varepsilon(0,\mu)=0,\qquad \mu>0 \tag{128}$$

$$u_\varepsilon(a,\mu)=u(a),\qquad \mu<0 \tag{129}$$

$$c(u-f)=\frac{\partial}{\partial x}\left(\frac{1}{3\Sigma(x)}\frac{\partial u}{\partial x}\right)+\gamma u\qquad \text{in}\quad]a,1[\tag{130}$$

$$u(1)=0 \tag{131}$$

$$\frac{1}{2}u(a)-\frac{\varepsilon}{3\Sigma}\frac{\partial u}{\partial x_i}(a)=\int_0^1 f_\varepsilon(a,\mu)\,\mu\,d\mu \tag{132}$$

Multiplying Eqs. (127) and (130) respectively by $\varphi_1 u_\varepsilon$, $\varphi_2 u_\varepsilon$, and $\varphi_3 u$ with φ_1, φ_2, and φ_3 positive functions to be precised later, integrating by parts, we obtain

We then obtain

$$\int_{-1}^{0}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_1 u_\varepsilon + \int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma - \gamma\right)\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\int_{-1}^{0}\frac{\mu}{2\varepsilon}u_\varepsilon^2(a,\mu)\,\varphi_1(a) - \int_{-1}^{0}\frac{\mu}{2\varepsilon}u_\varepsilon^2(0,\mu)\,\varphi_1(0) - \frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\int_{0}^{a} Ku_\varepsilon\varphi_1 u_\varepsilon = 0 \tag{133}$$

$$\int_{0}^{1}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_2 u_\varepsilon + \int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma - \gamma\right)\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\int_{0}^{1}\frac{\mu}{2\varepsilon}u_\varepsilon^2(a,\mu)\,\varphi_2(a) - \int_{0}^{1}\frac{\mu}{2\varepsilon}u_\varepsilon^2(0,\mu)\,\varphi_2(0) - \frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\int_{0}^{a} Ku_\varepsilon\varphi_2 u_\varepsilon = 0 \tag{134}$$

where $\alpha = \frac{1}{3\varepsilon}$.

Adding the boundary terms in (133) and (134), and using the boundary conditions, we obtain

$$\int_{-1}^{0}\left(\frac{\mu}{2\varepsilon}u_\varepsilon^2(a,\mu)\,\varphi_1(a) - \frac{\mu}{2\varepsilon}u_\varepsilon^2(0,\mu)\,\varphi_1(0)\right)$$
$$+\int_{0}^{1}\left(\frac{\mu}{2\varepsilon}u_\varepsilon^2(a,\mu)\,\varphi_2(a) - \frac{\mu}{2\varepsilon}u_\varepsilon^2(0,\mu)\,\varphi_2(0)\right) + \frac{\alpha}{2}\,[\varphi_3' u^2]_a^1 - \alpha\left[\varphi_3 u\frac{\partial u}{\partial x}\right]_a^1$$
$$= u^2(a)\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a) - \frac{1}{2\varepsilon}\int_{-1}^{0}\mu\varphi_1(0)\,u_\varepsilon^2(0,\mu)\,d\mu$$
$$+\frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu - \frac{\alpha}{2}\varphi_3'(a)\,u^2(a) + \alpha\varphi_3(a)\,u(a)\,u'(a)$$
$$= BC \tag{135}$$

Proceeding as in the previous sections, we obtain

$$BC \geqslant \left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a) - \frac{\alpha}{2}\varphi_3'(a)\right)u^2(a) - \frac{1}{\varepsilon}\varphi_3(a)\int_{0}^{1}\mu^2 f_\varepsilon^2(a,\mu)$$
$$-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu + \frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,u)\,\varphi_2(a)\,d\mu \tag{136}$$

Combining (133) and (134), we obtain

$$\int_{-1}^{0}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_1 u_\varepsilon + \int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma - \gamma\right)\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right) u\varepsilon^2\,dx\,d\mu$$

$$-\frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\int_{0}^{a} K u_\varepsilon \varphi_1 u_\varepsilon$$

$$+\int_{0}^{1}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_2 u_\varepsilon + \int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma - \gamma\right)\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right) u_\varepsilon^2\,dx\,d\mu$$

$$-\frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\int_{0}^{a} K u_\varepsilon \varphi_2 u_\varepsilon$$

$$+\int_{a}^{1} c(u-f)\,\varphi_3 u + \alpha\int_{a}^{1}\varphi_3 (u')^2\,dx - \int_{a}^{1}\left(\gamma\varphi_3 + \frac{\alpha}{2}\varphi_3''\right)u^2 + BC = 0 \tag{137}$$

Using the estimates of the collision integral terms obtained in the proof of the theorem about the large time behaviour, we obtain

$$\int_{-1}^{0}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_1 u_\varepsilon + \int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma - \gamma\right)\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right.$$

$$\left. -\frac{1}{\varepsilon^2}\Sigma\left[\int_{-1}^{0}\varphi_1\,d\mu' + \frac{1}{2}\varphi_1\right] - \frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\varphi_2\,d\mu'\right) u_\varepsilon^2\,dx\,d\mu$$

$$+\int_{0}^{1}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_2 u_\varepsilon + \int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{\varepsilon^2}\Sigma - \gamma\right)\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x} - \frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu'\right.$$

$$\left. -\frac{1}{\varepsilon^2}\Sigma\left[\int_{0}^{1}\varphi_2\,d\mu' + \frac{1}{2}\varphi_2\right]\right) u_\varepsilon^2\,dx\,d\mu$$

$$+\int_{0}^{a} c(u-f)\,\varphi_3 u + \alpha\int_{a}^{1}\varphi_3 (u')^2\,dx - \int_{a}^{1}\left(\gamma\varphi_3 + \frac{\alpha}{2}\varphi_3''\right)u^2 + BC \leqslant 0$$

Using (134), we obtain

$$\int_{-1}^{0}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_1 u_\varepsilon + \int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{2\varepsilon^2}\Sigma - \gamma\right)\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x} - \frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu' - \frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\varphi_2\,d\mu'\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\int_{0}^{1}\int_{0}^{a} c(u_\varepsilon - f_\varepsilon)\,\varphi_2 u_\varepsilon + \int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{2\varepsilon^2}\Sigma - \gamma\right)\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x} - \frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu' - \frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\varphi_2\,d\mu'\right)u_\varepsilon^2\,dx\,d\mu$$
$$+\int_{0}^{a} c(u-f)\,\varphi_3 u + \int_{a}^{1}\alpha\varphi_3 (u')^2\,dx - \int_{a}^{1}\left(\gamma\varphi_3 + \frac{\alpha}{2}\varphi_3''\right)u^2$$
$$+\left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a) - \frac{\alpha}{2}\varphi_3'(a)\right)u^2(a) - \frac{1}{\varepsilon}\varphi_3(a)\int_{0}^{1}\mu^2 f_\varepsilon^2(a,\mu)$$
$$-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu + \frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu \leqslant 0$$

Using Cauchy–Schwarz inequality, we obtain

$$\int_{-1}^{0}\int_{0}^{a}\left(\left(\frac{1}{2\varepsilon^2}\Sigma - \gamma + \frac{c}{2}\right)\varphi_1 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x} - \frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu' - \frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\varphi_2\,d\mu'\right)$$
$$\times u_\varepsilon^2\,dx\,d\mu + \int_{0}^{1}\int_{0}^{a}\left(\left(\frac{1}{2\varepsilon^2}\Sigma - \gamma + \frac{c}{2}\right)\varphi_2 - \frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x} - \frac{1}{\varepsilon^2}\Sigma\int_{-1}^{0}\varphi_1\,d\mu'\right.$$
$$\left.-\frac{1}{\varepsilon^2}\Sigma\int_{0}^{1}\varphi_2\,d\mu'\right)u_\varepsilon^2\,dx\,d\mu + \int_{0}^{1}\alpha\varphi_3 (u')^2\,dx + \int_{a}^{1}\left(\left(\frac{c}{2} - \gamma\right)\varphi_3 - \frac{\alpha}{2}\varphi_3''\right)u^2$$
$$+\left(\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a) - \frac{\alpha}{2}\varphi_3'(a)\right)u^2(a) - \frac{1}{\varepsilon}\varphi_3(a)\int_{0}^{1}\mu^2 f_\varepsilon^2(a,\mu)$$
$$-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu + \frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu$$
$$\leqslant \int_{-1}^{0}\int_{0}^{a}\frac{c}{2}\varphi_1 f_\varepsilon^2 + \int_{0}^{1}\int_{0}^{a}\frac{c}{2}\varphi_2 f_\varepsilon^2 + \int_{0}^{a}\frac{c}{2}\varphi_3 f^2 \qquad (138)$$

If we choose φ_1 and φ_2 such that

$$\int_{-1}^{0} \varphi_1(x, \mu)\, d\mu = 1$$

$$\int_{0}^{1} \varphi_2(x, \mu)\, d\mu = 1$$

we obtain

$$\begin{aligned}
&\int_{-1}^{0}\int_{0}^{a}\left(\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_1-2\right)+\left(-\gamma+\frac{c}{2}\right)\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu\\
&\quad+\int_{0}^{1}\int_{0}^{a}\left(\frac{1}{\varepsilon^2}\,\Sigma\left(\frac{1}{2}\varphi_2-2\right)+\left(-\gamma+\frac{c}{2}\right)\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}\right)u_\varepsilon^2\,dx\,d\mu\\
&\quad+\int_{a}^{1}\alpha\varphi_3(u')^2\,dx+\int_{a}^{1}\left(\left(\frac{c}{2}-\gamma\right)\varphi_3-\frac{\alpha}{2}\varphi_3''\right)u^2\\
&\quad+\left(\int_{-1}^{0}\frac{\mu}{\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\right)u^2(a)-\frac{1}{\varepsilon}\varphi_3(a)\int_{0}^{1}\mu^2 f_\varepsilon^2(a,\mu)\\
&\quad-\frac{1}{2\varepsilon}\int_{-1}^{0}\mu u_\varepsilon^2(0,\mu)\,\varphi_1(0)\,d\mu+\frac{1}{2\varepsilon}\int_{0}^{1}\mu u_\varepsilon^2(a,\mu)\,\varphi_2(a)\,d\mu\\
&\leqslant\int_{-1}^{0}\int_{0}^{a}\frac{c}{2}\varphi_1 f_\varepsilon^2+\int_{0}^{1}\int_{0}^{a}\frac{c}{2}\varphi_2 f_\varepsilon^2+\int_{0}^{a}\frac{c}{2}\varphi_3 f^2
\end{aligned}\tag{139}$$

Therefore, it becomes clear that we can choose φ_1, φ_2, and φ_3 positive functions bounded below and above by positive constants independent on μ and x, and satisfying

$$\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_1-2\right)-\gamma\varphi_1-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_1}{\partial x}=\delta\varphi_1$$

$$\frac{1}{\varepsilon^2}\Sigma\left(\frac{1}{2}\varphi_2-2\right)-\gamma\varphi_2-\frac{\mu}{2\varepsilon}\frac{\partial\varphi_2}{\partial x}=\delta\varphi_2$$

$$-\gamma\varphi_3-\frac{\alpha}{2}\varphi_3''=\delta\varphi_3$$

$$\int_{-1}^{0}\frac{\mu}{2\varepsilon}\varphi_1(a)-\frac{\alpha}{2}\varphi_3'(a)\geqslant 0$$

$$\frac{1}{2}\varphi_2(a)-\mu\varphi_3(a)\geqslant 0$$

Using the above construction of the functions φ_1, φ_2, and φ_3, we obtain

$$\|u_\varepsilon\|^2_{L^2}+\|u\|^2_{H^1}\to 0$$

This concludes the proof of our theorem. ∎

7. CONCLUSIONS

In this paper we proposed new models for the solution of transport equations in the case of intermediate regimes. We then have established the mathematical foundations of these new models. Our method consists of coupling the transport equations with their diffusion approximations. The transport equations are to be solved only in a small domain where the diffusion approximations are not accurate, while on most of the domain we solve the diffusion equations. The resulting method has the great advantage of using the correct model related to the physical features of the system. The proposed method gives an easy way of supplementing and testing a large variety of transport boundary conditions. Another advantage of the proposed method is that it can be applied in situations where the direct simulation of the transport equations is not possible (because of the lack of memory place and computer power. Because of the crucial importance of these new models in mathematical physics, we have developed their mathematical foundations. Our mathematical theory includes a rigorous derivation of the models the existence theory in L^∞ and positivity of the solutions to our models, the existence theory in L^2, the analysis of the asymptotic behaviour of the solutions, and the convergence of the transmission time marching algorithm applied to these new models. This mathematical theory is based in an essential way on the methods we introduced in refs. 14, 15. Other applications of these methods to problems of mathematical physics can be found in the work refs. 16, 17.

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